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Abstract
We address the question of radiation emission from mirrors following pre-scribed relativistic trajectories. The trajectories considered are asymptotically inertial: the mirror starts from rest and eventually reverts to motion at uniform velocity. This enables us to provide a description in terms of *in* and *out* states. We calculate exactly the Bogolubov α and β coefficients for a specific form of the trajectory, and stress the analytic properties of the amplitudes and the constraints imposed by unitarity. A formalism for the description of emission of radiation from a dispersive mirror is presented.

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1. Introduction

The question of emission of radiation from imperfect mirrors has received substantial attention in the last ten years. The present work is influenced by Barton and Eberlein (1993) where a Hamiltonian treatment of radiation from a dielectric mirror is presented, by Jaeckel and Reynaud (1993) and Barton and Calogeracos (1995) where a dispersive mirror is treated, and by Lambrecht *et al* (1996), Law (1995) and Schützhold *et al* (1998) where radiation from vibrating cavities is considered. The formalism has been exploited to study decoherence; see Maia Neto and Dalvit (2000). There are two common threads in the above works: (i) the treatments follow the Hamiltonian formulation of quantum field theories, (ii) the calculations are nonrelativistic (note however that the model presented in Barton and Calogeracos (1995) is manifestly covariant). This situation has to be contrasted with the study in the 1970s of radiation from relativistic perfect mirrors following prescribed trajectories (Davies and Fulling (1977) being one of the important early papers on the subject). In this connection we should note that (a) perfect mirrors have not yet been amenable to a Hamiltonian treatment and (b) the
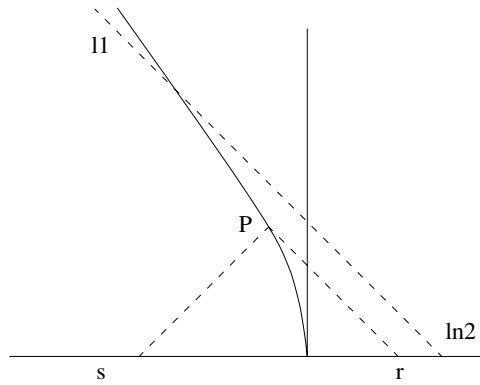
Figure 1. Asymptotically inertial trajectory starting from rest at O, accelerating from O to P and reverting to uniform acceleration at P.

treatment of perfect mirrors has traditionally focused on the relation to black hole radiation. As a result of (b) the trajectories considered were characterized by acceleration nonvanishing in the $t \to \infty$ limit. We examine the latter problem in a companion paper (Calogeracos 2002), hereafter referred to as II. Our objective here is to treat radiation from mirrors moving along prescribed relativistic asymptotically inertial trajectories. The approach we adopt in this paper may be of use in the calculation of the radiation emitted by the interface between two media when its speed approaches a characteristic speed of the system (cf the motion of the A–B interface in He3).

The present work, with the exception of appendix B, is devoted to perfect mirrors. The paper is organized as follows. In section 2, the normal modes for a one-sided perfect mirror are presented in the manner of Fulling and Davies (1976), Davies and Fulling (1977) and Birrell and Davies (1982). A physical interpretation of the modes in terms of the lab frame is given. In section 3, we consider a one-sided mirror that starts from rest, accelerates along the trajectory $z = -\ln(\cosh t)$ until it reaches a spacetime point P where it has some arbitrary velocity β_P, and then reverts to motion at uniform velocity (figure 1). The particular trajectory is of interest because it gives rise to a thermal spectrum if the acceleration is to go on forever. We present an exact calculation of $\beta(\omega, \omega')$ in terms of hypergeometric functions, and discuss the analytic properties of the $\alpha(\omega, \omega')$ and $\beta(\omega, \omega')$ amplitudes and the constraints imposed by unitarity. There are several advantages in considering asymptotically inertial trajectories: (i) acceleration continuing for an infinite time implies mathematical singularities and also entails physical pathologies associated, for example, with the infinite energy that has to be imparted to the mirror. Note that point P may lie as close to the null line $v = \ln 2$ as desired, or in other words the mirror's final speed may be close to the speed of light. (ii) The mirror's rest frame eventually (after acceleration stops) becomes an inertial frame and the standard description in terms of *in* and *out* states is possible. One may choose either the lab frame or the mirror's rest frame to describe the photons produced. (iii) One avoids statements about photons produced *while* the mirror is accelerated; rather one makes unambiguous statements pertaining to times $t = \pm\infty$ when acceleration vanishes. We show, in general, that for an asymptotically inertial trajectory with the velocity being continuous everywhere the amplitude squared $|\beta(\omega, \omega')|^2$ goes as $(\omega')^{-5}$ for large ω'. This is in contrast to a trajectory accelerated forever where $|\beta(\omega, \omega')|^2$ goes as $1/\omega'$ for large ω'. This radical difference is due to a subtle cancellation between two contributions, one arising from the initial and the other from the final asymptotic part of the trajectory; see II for details. In appendix A we give some mathematical details on
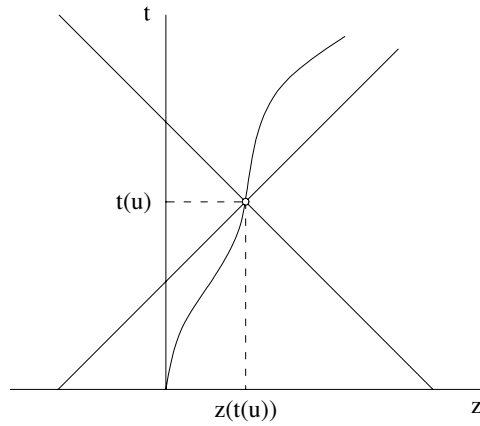
Figure 2. The spacetime coordinates t, z of a point on the trajectory as functions of u.

the trajectory. In appendix B we treat dispersive mirrors adopting the manifestly covariant approach of Barton and Calogeracos (1995). The dispersivity of the mirror is controlled by a parameter γ. We give the *in* and *out* modes for a dispersive mirror and outline how the method of section 3 may be applied to calculate the spectrum of the radiation emitted.

2. The normal modes

2.1. The standard treatment

Let us introduce the coordinates

$$u = t - z \qquad v = t + z. \tag{1}$$

The massless Klein–Gordon equation reads

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0. \tag{2}$$

Hence any function that depends only on u or v (or the sum of two such functions) is a solution of (2). Let $z = g(t)$ be the mirror's trajectory. We take everything to exist to the right of the mirror. The objective is to find a set of modes satisfying (2) and the boundary condition

$$\phi(t, g(t)) = 0. \tag{3}$$

One set of modes (Birrell and Davies (1982), equation (4.43)) is

$$\varphi_\omega(u, v) = \frac{i}{2\sqrt{\pi \omega}}(\exp(-i\omega v) - \exp(-i\omega p(u))). \tag{4}$$

We will always consider (4) and (21) below as functions of the lab coordinates (t, z) through (1). The determination of the function $p(u)$ goes as follows. For a certain value of u consider the point where the line $u = $ constant cuts the mirror's trajectory (figure 2). Let $t(u)$ be the time coordinate of that point. Define

$$p(u) = 2t(u) - u. \tag{5}$$

Since by (1)

$$t(u) - u = g(t(u)) \tag{6}$$

it follows that

$$p(u) = t(u) + g(t(u)). \tag{7}$$

From the second part of (1) it follows that

$$v = p(u). \tag{8}$$

It is clear from (8) that $\varphi_\omega(u, v)$ vanishes on the trajectory. Equation (8) describes the trajectory in terms of u, v variables. Conversely

$$u = f(v) \tag{9}$$

where the function f is the inverse of p. The graphical interpretation is that an outgoing (rightward) light ray corresponding to a certain u cuts the trajectory at a point where the incoming (leftward) light ray is labelled by $v = p(u)$. From definitions (1) and (8) it follows that for points (t, z) on the trajectory

$$t + z = p(t - z). \tag{10}$$

The following observation will be of use later on. Consider motion along the trajectory $-g(t)$, i.e. the reflection of the trajectory $g(t)$ with respect to the origin. It is clear from (10) that the equations for the two trajectories are related via the correspondence $u \rightleftharpoons v$, $f \rightleftharpoons p$.

We introduce the velocity

$$\beta = \frac{\mathrm{d}g}{\mathrm{d}t}.$$

Differentiating (10) with respect to time we obtain

$$p'(u) = \frac{1 + \beta(u)}{1 - \beta(u)} \tag{11}$$

where β stands for the instantaneous velocity (not to be confused with the amplitude $\beta(\omega, \omega')$; the latter will always be a function of two variables). From the fact that f and p are inverse it follows that

$$f'(v) = \frac{1 - \beta(v)}{1 + \beta(v)}. \tag{12}$$

For motion at uniform velocity B, the above expression yields

$$f_0(v) = C + \frac{1 - B}{1 + B}v \tag{13}$$

where the constant C is related to the initial condition. From (11) and (12) we may readily obtain β along the trajectory as a function of either u or v:

$$\beta(u) = \frac{p'(u) - 1}{p'(u) + 1} \tag{14}$$

$$\beta(v) = \frac{1 - f'(v)}{1 + f'(v)}. \tag{15}$$

In the case of the accelerating trajectory $g(t) = -\ln(\cosh t)$ considered in section 3 the function $f(v)$ is denoted by f_{acc} and given by (78) (with $\kappa = 1$):

$$f_{\mathrm{acc}}(v) = -\ln(2 - \mathrm{e}^v). \tag{16}$$

The quantity $\mathrm{d}\beta/\mathrm{d}u$ will turn up in the treatment of dispersive mirrors. We write

$$\frac{\mathrm{d}\beta}{\mathrm{d}u} = \frac{\mathrm{d}\beta}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}u} \tag{17}$$

and from the definition of u we get

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 1 - \beta. \tag{18}$$

Combination of (17) and (18) yields

$$\frac{\mathrm{d}\beta}{\mathrm{d}u} = \frac{a}{1 - \beta} \tag{19}$$

where a stands for the acceleration $\mathrm{d}\beta/\mathrm{d}t$. Similarly we obtain

$$\frac{\mathrm{d}\beta}{\mathrm{d}v} = \frac{a}{1 + \beta}. \tag{20}$$

Note that another set of modes satisfying the boundary condition is immediately obtained from (4)

$$\bar{\varphi}_\omega(u, v) = \frac{\mathrm{i}}{2\sqrt{\pi\omega}}(\exp(-\mathrm{i}\omega f(v)) - \exp(-\mathrm{i}\omega u)). \tag{21}$$

We quote the time derivative for future reference

$$\frac{\partial\bar{\varphi}_\omega(u, v)}{\partial t} = \frac{\omega}{2\sqrt{\pi\omega}}(f'(v)\exp(-\mathrm{i}\omega f(v)) - \exp(-\mathrm{i}\omega u)). \tag{22}$$

The modes $\varphi_\omega(u, v)$ of (4) describe waves incident from the right as is clear from the sign of the exponential in the first term; the second term represents the reflected part which has a rather complicated behaviour depending on the motion of the mirror. These modes constitute the *in* space and should obviously be unoccupied before acceleration starts,

$$a_i|0_{\mathrm{in}}\rangle = 0$$

according to the following section. Similarly, $\bar{\varphi}_\omega(u, v)$ describe the modes travelling to the right (emitted by the mirror) as can be seen from the exponential of the second term. Correspondingly the first term is complicated. These modes define the *out* space and

$$\bar{a}_i|0_{\mathrm{out}}\rangle = 0.$$

The state $|0_{\mathrm{out}}\rangle$ corresponds to the state where nothing is produced by the mirror. The two representations are connected by the Bogolubov transformation to be reviewed in the following section. The fact that the *in* and *out* vacua are not identical lies at the origin of particle production.

2.2. An illustration

It is instructive to consider motion at uniform velocity

$$z = Bt. \tag{23}$$

The function f is given in this case by (13) with $C = 0$ (since the condition $f(0) = 0$ is valid for the trajectory (23))

$$f(v) = v\frac{1 - B}{1 + B}. \tag{24}$$

Thus mode (4) reads

$$\varphi_\omega(u, v) = \frac{\mathrm{i}}{2\sqrt{\pi\omega}}\left(\exp(-\mathrm{i}\omega v) - \exp\left(-\mathrm{i}\omega u\frac{1 + B}{1 - B}\right)\right). \tag{25}$$

On the other hand, in the case of uniform velocity the modes may be obtained by simply boosting the modes pertaining to the stationary mirror. We denote the coordinates in the frame

of the mirror by (t', z') and by Ω' and K' the energy and momentum, respectively, of a mode in the stated frame (of course $\Omega' = |K'|$ but we keep them distinct for the moment for the sake of clarity). The mode satisfying the boundary condition at the position of the mirror, taken to be at the origin $z' = 0$, reads (modulo a normalization factor)

$$\exp(-i\Omega't' - iK'z') - \exp(-i\Omega't' + iK'z') \tag{26}$$

where the first term above refers to the incident and the second to the reflected wave. We express the comoving coordinates in terms of the lab frame coordinates via

$$z' = \frac{z - Bt}{\sqrt{1 - B^2}} \qquad t' = \frac{t - Bz}{\sqrt{1 - B^2}}$$

and substitute in (26) to get the mode in the form

$$\exp\left(-i\frac{\Omega' - K'B}{\sqrt{1 - B^2}}t + i\frac{-K' + \Omega'B}{\sqrt{1 - B^2}}z\right) - \exp\left(-i\frac{\Omega' + K'B}{\sqrt{1 - B^2}}t + i\frac{K' + \Omega'B}{\sqrt{1 - B^2}}z\right).$$

Using $\Omega' = |K'|$ we rewrite the above as

$$\exp\left(-i\frac{\Omega'(1 - B)}{\sqrt{1 - B^2}}(t + z)\right) - \exp\left(-i\frac{\Omega'(1 + B)}{\sqrt{1 - B^2}}(t - z)\right). \tag{27}$$

Observe that expression (27) is identical to (21) if we set

$$\omega = \frac{\Omega'(1 + B)}{\sqrt{1 - B^2}} = \Omega'\sqrt{\frac{1 + B}{1 - B}} \tag{28}$$

and use (24). The second term in (27) makes it clear that the ω appearing in (21) is the photon energy in the lab frame.

3. Radiation from a perfect mirror accelerated for a finite time

3.1. The Bogolubov transformation

In this subsection, we fix conventions and notation following Birrell and Davies (1982), section 3.2, write down the general expression for the Bogolubov amplitude $\beta(\omega, \omega')$ to be evaluated, and study its asymptotic behaviour for large ω'. The creation and annihilation operators referring to the two sets $\varphi, \bar{\varphi}$ are connected by

$$\bar{a}_i = \sum_j (\alpha_{ji}a_j + \beta_{ji}^* a_j^\dagger). \tag{29}$$

Using (29) and its Hermitian conjugate we may immediately verify that the expectation value of the number of excitations of the mode (i) in the $|0_{\text{in}}\rangle$ vacuum is given by

$$\langle 0_{\text{in}}|\bar{N}_i|0_{\text{in}}\rangle = \sum_j |\beta_{ji}|^2. \tag{30}$$

In our notation the matrix β_{ji} is given by the overlap

$$\beta(\omega, \omega') = -\langle \bar{\varphi}_\omega, \varphi_{\omega'}^* \rangle \tag{31}$$

defined as (see Birrell and Davies (1982), equations (2.9) and (3.36))

$$\beta(\omega, \omega') = -i \int_0^\infty dz\, \varphi_{\omega'}(z, 0)\frac{\partial}{\partial t}\bar{\varphi}_\omega(z, 0) + i \int_0^\infty dz \left(\frac{\partial}{\partial t}\varphi_{\omega'}(z, 0)\right)\bar{\varphi}_\omega(z, 0). \tag{32}$$

The integration in (32) can be over any spacelike hypersurface. In all cases examined here the mirror is at rest for $t \leqslant 0$ and the choice $t = 0$ for the hypersurface is thus convenient. Note the technical simplification that such a trajectory offers. The *in* modes evaluated at $t = 0$

involve the function $p(-z)$ (i.e. p evaluated at negative values of the argument), and are thus given by the simple expression (25) evaluated at $B = 0$. The $\bar{\varphi}$ modes are given by (21) with f depending on the trajectory. Relation (32) is rewritten more explicitly in the form (f is a function of z and its functional form is determined by (12) and the initial condition)

$$
\begin{aligned}
\beta(\omega, \omega') = &-\mathrm{i} \int_0^\infty \mathrm{d}z \, \frac{\mathrm{i}}{2\sqrt{\pi \omega'}} \{\mathrm{e}^{-\mathrm{i}\omega'z} - \mathrm{e}^{\mathrm{i}\omega'z}\} \frac{\omega}{2\sqrt{\pi \omega}} \{f' \mathrm{e}^{-\mathrm{i}\omega f} - \mathrm{e}^{\mathrm{i}\omega z}\} \\
&+\mathrm{i} \int_0^\infty \mathrm{d}z \, \frac{\omega'}{2\sqrt{\pi \omega'}} \{\mathrm{e}^{-\mathrm{i}\omega'z} - \mathrm{e}^{\mathrm{i}\omega'z}\} \frac{\mathrm{i}}{2\sqrt{\pi \omega}} \{\mathrm{e}^{-\mathrm{i}\omega f} - \mathrm{e}^{\mathrm{i}\omega z}\}.
\end{aligned}
\tag{33}
$$

The above expression may be rearranged in the form

$$
\begin{aligned}
\beta(\omega, \omega') = &\frac{\mathrm{i}\omega'}{2\pi\sqrt{\omega\omega'}} \int_0^\infty \mathrm{d}z \, \sin(\omega'z) \{\mathrm{e}^{-\mathrm{i}\omega f} - \mathrm{e}^{\mathrm{i}\omega z}\} \\
&+\frac{\mathrm{i}\omega}{2\pi\sqrt{\omega\omega'}} \int_0^\infty \mathrm{d}z \, \sin(\omega'z) \{\mathrm{e}^{\mathrm{i}\omega z} - f' \mathrm{e}^{-\mathrm{i}\omega f}\}.
\end{aligned}
\tag{34}
$$

Note that both (33) and (34) are direct descendants of (32). Note also the presence of two f-independent (i.e. trajectory-independent) terms whose origin is purely kinematic. The amplitude $\beta(\omega, \omega')$ will be evaluated for a specific trajectory in the next subsection.

In this section we confine ourselves to trajectories that in the infinite past and infinite future describe motion at uniform velocity. In that case both *in* and *out* solutions form complete sets of states, and the standard manipulations with creation and annihilation operators go through. One problem that arises in connection with completeness is the following. The wave equation is solved by specifying Cauchy data along the (spacelike) line $t = 0$, $z > 0$ and the Dirichlet boundary condition along the (timelike) curve that corresponds to the trajectory of the mirror. In the case of an asymptotically inertial trajectory the two curves along which the boundary conditions are specified cut the $u = t - z$, $v = t + z$ characteristics of the wave equation (which is the normal case). Things change if the mirror starts from rest at $t = 0$, $z = 0$ and follows the trajectory (studied in the next subsection) $g(t) = -\ln(\cosh t)$ forever, thus asymptotically approaching the null line $v = -\ln 2$. Then the trajectory does not cut all the characteristics of the wave equation (it leaves out those of $v = $ const lying above the null line $v = \ln 2$). Hence the boundary conditions allow for an arbitrary solution $\varphi(v)\theta(v - \ln 2)$, which should be added to the usual expansion of the field in terms of the *out* modes $\bar{\varphi}$ and the *out* creation and annihilation operators $\bar{a}_i^\dagger$, $\bar{a}_i$; see Nikishov and Ritus (1995).

The number density per unit frequency ω of the emitted photons is obtained after we integrate the modulus squared of (34) over ω' (the integration over ω' is the analogue of the summation over j in (30)):

$$
\frac{\mathrm{d}N(\omega)}{\mathrm{d}\omega} = \int_0^\infty \mathrm{d}\omega' \, |\beta(\omega, \omega')|^2 \, \mathrm{e}^{-\alpha(\omega+\omega')}.
\tag{35}
$$

The small convergence factor α has been introduced in anticipation of an ultraviolet divergence in (35). The fact that quantities such as the total number of particles produced or the total emitted energy suffer from such divergences is also encountered in nonrelativistic calculations in the case of dispersive mirrors (see, for example, Barton and Calogeracos (1995)). The total emitted energy is given by

$$
E = \int_0^\infty \mathrm{d}\omega \, \omega \int_0^\infty \mathrm{d}\omega' \, |\beta(\omega, \omega')|^2 \, \mathrm{e}^{-\alpha(\omega+\omega')}.
\tag{36}
$$

The behaviour of the integrand in (35) for large ω' is crucial. As mentioned in the introduction we are interested in asymptotically inertial trajectories defined as follows. The mirror is stationary at O for $t < 0$, follows an accelerating trajectory until some spacetime point P, and then continues at uniform velocity β_P. We take the velocity to be continuous throughout (points O and P included). Such a trajectory is shown in figure 1 and will be thoroughly examined in the next subsection. Let r be the z intercept of the null line passing through P with slope -1. Walker (1985) has asserted that in the case of a trajectory with continuous velocity the square $|\beta(\omega, \omega')|^2$ goes as $(\omega')^{-5}$. We present a proof based on the technique of Lighthill (1958), ch 4. We introduce the two integrals

$$J_1(\omega') \equiv \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\omega' z} \left\{ \mathrm{e}^{-\mathrm{i}\omega f(z)} - \mathrm{e}^{\mathrm{i}\omega z} \right\} \tag{37}$$

$$J_2(\omega') \equiv \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\omega' z} \left\{ f'(z)\, \mathrm{e}^{-\mathrm{i}\omega f(z)} - \mathrm{e}^{\mathrm{i}\omega z} \right\}. \tag{38}$$

Then the first and second integrals appearing in (34) are proportional to

$$J_1(-\omega') - J_1(\omega') \tag{39}$$

and

$$J_2(-\omega') - J_2(\omega') \tag{40}$$

respectively. We now examine the asymptotic estimates for large ω' of $J_1(\omega')$ and $J_2(\omega')$.

We define the functions

$$\begin{aligned}
\Psi_A(z) &\equiv \exp(-\mathrm{i}\omega f_{\mathrm{in}}(z)) - \exp(\mathrm{i}\omega z) \\
\Psi_B(z) &\equiv \exp(-\mathrm{i}\omega f_{\mathrm{acc}}(z)) - \exp(\mathrm{i}\omega z) \\
\Psi_C(z) &\equiv \exp(-\mathrm{i}\omega f_0(z)) - \exp(\mathrm{i}\omega z)
\end{aligned} \tag{41}$$

where the functions $f_{\mathrm{in}}, f_{\mathrm{acc}}, f_0$ are defined by (13) for $B = 0, C = 0$, (16) and (13) for velocity equal to β_P, respectively. In the latter case C is fixed by the requirement that the velocity be continuous at P (see (49)). To conform with Lighthill's notation we define $\omega \equiv 2\pi y$. Then $J_1(\omega')$ is the Fourier transform $G(y)$ of

$$\Psi(z) \equiv \Psi_A(z)H(-z) + \Psi_B(z)H(z) + [\Psi_C(z) - \Psi_B(z) - \Psi_A(z)]\, H(z - r) \tag{42}$$

where H is the Heaviside unit function. The singular points of $\Psi(z)$ are $z = 0$ and $z = r$. Clearly, the three functions $\Psi_A(z), \Psi_B(z), \Psi_C(z)$ correspond to the three stages of a stationary mirror, accelerating mirror and mirror moving at uniform velocity, respectively. Observe that the general connection between the velocity β and the function f' is given by (12) and that the velocity is continuous everywhere along the trajectory considered whereas the acceleration has different (but finite) left and right derivatives at each of the two singular points. Thus β'' (the *second* time derivative of the velocity) has a finite jump at each of the two singular points and hence is absolutely integrable everywhere. To treat the first singular point, let us define the function

$$\begin{aligned}
F_1(z) = {}& \left[\Psi_A(0) + z\Psi_A'(0) + \frac{z^2}{2!}\Psi_A''(0) + \frac{z^3}{3!}\Psi_A'''(0) \right] H(-z) \\
& + \left[\Psi_B(0) + z\Psi_B'(0) + \frac{z^2}{2!}\Psi_B''(0) + \frac{z^3}{3!}\Psi_A'''(0) \right] H(z).
\end{aligned} \tag{43}$$

According to what was said above, the quantity $\Psi_A'''(0) - \Psi_B'''(0)$ is nonvanishing but finite. Hence the function $\Psi(z) - F_1(z)$ has absolutely integrable third derivative at $z = 0$. We treat the second singular point $z = r$ by a similar method, defining the function

$$F_2(z) = \left[\Psi_B(r) + (r - z)\Psi_B'(r) + \frac{(r - z)^2}{2!}\Psi_B''(r) + \frac{(r - z)^3}{3!}\Psi_B'''(r) \right] H(r - z)$$
$$+ \left[\Psi_C(r) + (z - r)\Psi_C'(r) + \frac{(z - r)^2}{2!}\Psi_C''(0) + \frac{(z - r)^3}{3!}\Psi_C'''(0) \right] H(z - r).$$
$$(44)$$

As before, the quantity $\Psi_C'''(r) - \Psi_B'''(r)$ is nonvanishing but finite. Finally, observe that the function $\Psi'''(z)$ is well behaved at infinity in the sense of Lighthill's (1958) definition 20, p 49. Then according to Lighthill's theorem 19, p 52, the Fourier transform $G(y)$ of $\Psi(z)$ takes the form
$$G(y) = G_1(y) + G_2(y) + o(|y|^{-3}) \qquad (45)$$
where $G_1(y)$, $G_2(y)$ are the Fourier transforms of $F_1(z)$, $F_2(z)$. According to Lighthill's table, p 43, the functions $G_1(y)$, $G_2(y)$ go asymptotically as y^{-4}. Hence overall $G(y) \approx o(|y|^{-3})$. Returning to (37) we observe that $J_1(-\omega')$ goes as $(\omega')^{-3}$ and so does the difference (39). Regarding integral (38) we may repeat the above process almost verbatim. The one difference is that this integral involves f', thus in the previous argument one has to substitute 'second derivative' for 'third derivative' and accordingly keep terms of second order rather than third in (43) and (44). Hence the second integral goes as $(\omega')^{-2}$. Noting the prefactors in (34) and especially the fact that the first integral is multiplied by one power of ω' we deduce that overall $|\beta(\omega, \omega')|^2$ indeed goes as $(\omega')^{-5}$. It is shown in section 3 of II that the above arguments cannot be applied in the case of a mirror accelerating forever and that in that case $|\beta(\omega, \omega')|^2$ behaves as $1/\omega'$.

3.2. The Bogolubov amplitudes for a mirror eventually reverting to uniform velocity

We remind the reader that the type of asymptotically inertial trajectory we are interested in is defined after equation (36) (see figure 1). The part of the trajectory between O and P is given by
$$z = g(t) = -\ln(\cosh t). \qquad (46)$$
The velocity is continuous at O and P whereas of course the acceleration $a = \mathrm{d}\beta/\mathrm{d}t$ is not. The trajectory equation (46) is a special case of $z = -\frac{1}{\kappa}\ln(\cosh(\kappa t))$. The energy scale is fixed in (46) by setting $\kappa = 1$. Our interest in the trajectory (46) stems from the fact that if (and only if) the acceleration is allowed to go on forever then the mirror gives rise to a thermal spectrum of radiation. For details see II and references therein. Note that for large t (46) has the asymptotic form
$$g(t) \approx -t - \mathrm{e}^{-2t} + \ln 2. \qquad (47)$$
The advantages of an asymptotically inertial trajectory have been mentioned earlier on.

The quantity r was defined in the previous subsection as the z intercept of the null line passing through P with slope -1. As can be seen from the equation of the trajectory $v = p(u)$ with $p(u)$ given by (76) the possible values of r range from 0 to $\ln 2$ (the latter asymptotically when $u \to \infty$). The quantities r and β_P are connected via (8)
$$\beta_P = 1 - \mathrm{e}^r.$$

When it comes to the integrations appearing in the amplitude (33), we use for $f(z)$ the expression $f_{\mathrm{acc}}(z)$ given by (16) in the range $0 \leqslant z \leqslant r$ and $f_0(z)$ given by (13) in the range

$r \leqslant z \leqslant \infty$. Some of the integrals involved are conveniently expressed in terms of the function ζ and its complex conjugate ζ^* defined in Heitler (1954), pp 66–71:

$$\zeta(x) \equiv -\mathrm{i} \int_0^\infty \mathrm{e}^{\mathrm{i}\kappa x}\, \mathrm{d}\kappa = P\frac{1}{x} - \mathrm{i}\pi\delta(x). \tag{48}$$

The constant C in (13) is determined by continuity of f at the point P

$$f_{\mathrm{acc}}(z) = f_0(z)$$

or

$$C + \frac{1 - \beta_P}{1 + \beta_P}r = -\ln(2 - \mathrm{e}^r) \tag{49}$$

where we used (78) for the right-hand side of (49).

Amplitude (34) is split into three contributions

$$\beta(\omega, \omega') \equiv \beta_I(\omega, \omega') + \beta_{II}(\omega, \omega') + \beta_{III}(\omega, \omega') \tag{50}$$

originating as follows. The $\beta_{III}(\omega, \omega')$ stands for the contribution of the two terms in (34) that are f-independent, is always there for a mirror starting from rest (or initially moves at uniform velocity) and does not depend on the subsequent form of the trajectory. The $\beta_I(\omega, \omega')$ contribution results from the accelerating part of the trajectory and corresponds to the $0 < z < r$ integration range in (34). Its evaluation will prove mathematically somewhat involved. The $\beta_{II}(\omega, \omega')$ amplitude results from the $z > r$ part of the integration range in (34), is readily evaluated and is specific to a mirror that reverts to the state of uniform motion. Thus

$$\beta_I(\omega, \omega') = -\frac{\mathrm{i}}{2\pi\sqrt{\omega'\omega}} \int_0^r \mathrm{d}z\, \sin(\omega'z)\{\omega f'_{\mathrm{acc}}(z) - \omega'\}\, \mathrm{e}^{-\mathrm{i}\omega f_{\mathrm{acc}}(z)} \tag{51}$$

$$\beta_{II}(\omega, \omega') = \frac{1}{4\pi\mathrm{i}\sqrt{\omega\omega'}}\left(\omega\frac{1 - \beta_P}{1 + \beta_P} - \omega'\right)\exp\left[\mathrm{i}\left(\omega\frac{1 - \beta_P}{1 + \beta_P} + \omega'\right)r\right]$$
$$\times \zeta\left(\omega\frac{1 - \beta_P}{1 + \beta_P} + \omega'\right)\mathrm{e}^{-\mathrm{i}\omega C} - \frac{1}{4\pi\mathrm{i}\sqrt{\omega\omega'}}\exp\left[\mathrm{i}\left(-\omega' + \omega\frac{1 - \beta_P}{1 + \beta_P}\right)r\right]\mathrm{e}^{-\mathrm{i}\omega C} \tag{52}$$

$$\beta_{III}(\omega, \omega') = \frac{1}{4\pi\mathrm{i}\sqrt{\omega\omega'}} - \frac{1}{4\pi\mathrm{i}\sqrt{\omega\omega'}}(\omega - \omega')\zeta(\omega + \omega'). \tag{53}$$

Note that the arguments of the ζ functions in (52) and (53) never vanish. Thus as far as the evaluation of $\beta(\omega, \omega')$ is concerned we observe that it is only the first term in (48) that is operative, and that $\beta(\omega, \omega')$ does not have any δ-type singularities. However, the forms (52) and (53) are useful because we can obtain the other Bogolubov amplitude $\alpha(\omega, \omega')$ via the substitution $\omega \to -\omega$ in accordance with (72) below. The $\alpha(\omega, \omega')$ is of course expected to have δ-type singularities, and in fact reduces to just $\delta(\omega - \omega')$ in the trivial case when the *in* and *out* modes coincide.

We turn our attention to $\beta_I(\omega, \omega')$. We integrate (51) by parts to get

$$\beta_I(\omega, \omega') = \frac{1}{2\pi\sqrt{\omega\omega'}}\sin(\omega'r)\,\mathrm{e}^{-\mathrm{i}\omega f_{\mathrm{acc}}(r)} - \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_0^r \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\omega f_{\mathrm{acc}}(z) - \mathrm{i}\omega'z} \tag{54}$$

where the first term is the upper endpoint contribution; the lower endpoint contribution vanishes. We rewrite (54) using (16) for f_{acc} as

$$\beta_I(\omega, \omega') = \frac{1}{2\pi\sqrt{\omega\omega'}}\sin(\omega'r)(2 - \mathrm{e}^r)^{\mathrm{i}\omega} - \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_0^r \mathrm{d}z\, (2 - \mathrm{e}^z)^{\mathrm{i}\omega}\,\mathrm{e}^{-\mathrm{i}\omega'z}. \tag{55}$$

We make the change of variable

$$z = \ln 2 - \rho \tag{56}$$

and rewrite (55) in the form

$$\beta_I(\omega, \omega') = \frac{1}{2\pi \sqrt{\omega\omega'}} \sin(\omega' r)(2 - e^r)^{i\omega} - \frac{2^{i(\omega-\omega')}}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_{\ln 2-r}^{\ln 2} d\rho \, (1 - e^{-\rho})^{i\omega} e^{i\omega'\rho}. \tag{57}$$

We isolate the integral occurring in (57) and rewrite it with arbitrary limits:

$$I \equiv \int_{l_1}^{l_2} d\rho (1 - e^{-\rho})^{i\omega} e^{i\omega'\rho}. \tag{58}$$

We use the binomial expansion in (58) and integrate term by term to get

$$I = \left[-e^{i\omega'x} \sum_{n=0}^{\infty} \frac{(-i\omega)_n}{n!} \frac{(e^{-x})^n}{(n - i\omega')} \right]_{x=l_1}^{x=l_2} \tag{59}$$

where $(\cdot)_n$ is the Pochhammer symbol. We write

$$\frac{1}{(n - i\omega')} = \frac{(-i\omega')_n}{(-i\omega' + 1)_n} \tag{60}$$

and then the sum in (59) is identified as the F hypergeometric function:

$$I = \left[\frac{e^{i\omega'x}}{i\omega'} F(-i\omega, -i\omega'; -i\omega' + 1; e^{-x}) \right]_{x=l_1}^{x=l_2}. \tag{61}$$

Note that according to item 15.1.1 (b) of Abramowitz and Stegun (1972) the series expansion of the $F(a, b, c, z)$ hypergeometric converges absolutely if $\mathrm{Re}(c - a - b) = 1 > 0$ and this inequality does hold for the hypergeometric occurring in (61). We reinstate the limits $l_1 = \ln 2 - r$, $l = \ln 2$ appearing in (57) and obtain for the $\beta_I(\omega, \omega')$ amplitude the expression

$$\beta_I(\omega, \omega') = \frac{1}{2\pi \sqrt{\omega\omega'}} \sin(\omega' r)(2 - e^r)^{i\omega} - \frac{1}{2\pi i} \frac{2^{i\omega}}{\sqrt{\omega\omega'}} \left\{ F\left(-i\omega, -i\omega'; -i\omega' + 1; \frac{1}{2}\right) \right.$$

$$\left. + F\left(-i\omega, -i\omega'; -i\omega' + 1; \frac{e^r}{2}\right) e^{-i\omega'r} \right\}. \tag{62}$$

It is interesting to observe that the value $r = \ln 2$ corresponding to the asymptote falls exactly on the radius of convergence of the series as one can see from the argument of the second hypergeometric appearing above. (Note that according to (80) $e^r = 1 - \beta_P$, recall that β_P is the velocity at P, and that asymptotically $\beta \to -1$.)

An alternative way to evaluate (58) is to rewrite the integral occurring in (57)

$$I = \int_{\ln 2-r}^{\ln 2} d\rho \, (1 - e^{-\rho})^{i\omega} e^{i\omega'\rho - \alpha\rho} \tag{63}$$

introducing a small positive constant α to ensure convergence. We change variable to

$$t = e^{-\rho}.$$

Then (63) reads

$$I = \int_{1/2}^{e^r/2} dt \, t^{-1-i\omega'+\alpha}(1 - t)^{i\omega} = \int_0^{e^r/2} dt \, t^{-1-i\omega'+\alpha}(1 - t)^{i\omega} - \int_0^{1/2} dt \, t^{-1-i\omega'+\alpha}(1 - t)^{i\omega}.$$

If we make use of the integral representation 15.3.1 of Abramowitz and Stegun (1972) for the hypergeometric function and take the $\alpha \to 0$ limit, we get result (61) for I.

Relation (62) provides the final result for $\beta_I(\omega, \omega')$. However, for a numerical evaluation (via MAPLE for example) of the second hypergeometric

$$F\left(-i\omega, -i\omega'; -i\omega' + 1; \frac{e^r}{2}\right) \tag{64}$$

occurring in (62) and if the point P is near the asymptote (i.e. $e^r/2 \simeq 1$) one can accelerate the convergence through the use of the linear transformation formula 15.3.6 of Abramowitz and Stegun (1972)

$$F(a, b; c; z) = F(a, b; a + b - c + 1; 1 - z)\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}$$
$$+ F(c - a, c - b; c - a - b + 1; 1 - z)(1 - z)^{c - a - b}\frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}. \tag{65}$$

Using our a, b, c the first hypergeometric in (65) reads $F(-i\omega, -i\omega'; -i\omega; 1 - e^r/2)$. We expand it in the standard hypergeometric series and for this particular set of parameters the series reduces to the simple binomial expansion and get

$$F(-i\omega, -i\omega'; -i\omega; 1 - e^r/2) = \left(\frac{e^r}{2}\right)^{\omega'}.$$

We may thus rewrite the amplitude (62)

$$\beta_I(\omega, \omega') = \frac{1}{2\pi\sqrt{\omega\omega'}}\sin(\omega' r)(2 - e^r)^{i\omega} - \frac{1}{2\pi i}\frac{2^{i\omega}}{\sqrt{\omega\omega'}}\left\{F\left(-i\omega, -i\omega'; -i\omega' + 1; \frac{1}{2}\right)\right.$$
$$+ \frac{\Gamma(-i\omega' + 1)\Gamma(i\omega + 1)}{\Gamma(1 - i\omega' + i\omega)}\left(\frac{e^r}{2}\right)^{\omega'}e^{-i\omega' r}$$
$$+ F\left(-i\omega' + 1 + i\omega, 1; i\omega + 2; 1 - \frac{e^r}{2}\right)\frac{\Gamma(-i\omega' + 1)\Gamma(-i\omega - 1)}{\Gamma(-i\omega')\Gamma(-i\omega)}$$
$$\times \left.\left(1 - \frac{e^r}{2}\right)^{1 + i\omega}e^{-i\omega' r}\right\}. \tag{66}$$

To summarize, the advantage of the above form over (62) lies in the fact that for a point P near the asymptotic line the arguments of the hypergeometrics are far from unity (the radius of convergence). In conclusion the sum of (52), (53) and (66) is the exact answer for the $\beta(\omega, \omega')$ amplitude for the trajectory in question.

3.3. On the Bogolubov coefficients

The Bogolubov α coefficients appearing in (29) are given by the analogue of (31),

$$\alpha(\omega, \omega') = \langle \bar{\varphi}_\omega, \varphi_{\omega'} \rangle \tag{67}$$

or explicitly by

$$\alpha(\omega, \omega') = i\int_0^\infty dz\, \varphi_{\omega'}(z, 0)\frac{\partial}{\partial t}\bar{\varphi}_\omega^*(z, 0) - i\int_0^\infty dz\left(\frac{\partial}{\partial t}\varphi_{\omega'}(z, 0)\right)\bar{\varphi}_\omega^*(z, 0). \tag{68}$$

Recall also the unitarity condition (3.39) of Birrell and Davies (1982),

$$\int_0^\infty d\tilde{\omega}\,(\alpha(\tilde{\omega}, \omega_1)\alpha^*(\tilde{\omega}, \omega_2) - \beta(\tilde{\omega}, \omega_1)\beta^*(\tilde{\omega}, \omega_2)) = \delta(\omega_1 - \omega_2) \tag{69}$$

and its partner

$$\int_0^\infty d\tilde{\omega}\,(\alpha(\omega_1,\tilde{\omega})\alpha^*(\omega_2,\tilde{\omega}) - \beta(\omega_1,\tilde{\omega})\beta^*(\omega_2,\tilde{\omega})) = \delta(\omega_1 - \omega_2). \qquad (70)$$

Relations (69) and (70) above are direct consequences of the fact that the sets $\bar{\varphi}_\omega$ and $\varphi_{\omega'}$, respectively, are orthonormal and complete. They also guarantee that the operators a_j, $a_j^\dagger$ and $\bar{a}_i$, $\bar{a}_i^\dagger$ obey the standard equal time commutation relations that creation and annihilation operators do.

We find it convenient to isolate the square roots in (34) and introduce quantities $A(\omega,\omega')$, $B(\omega,\omega')$ that are analytic functions of the frequencies (without the branch cuts attendant to square roots) via

$$\alpha(\omega,\omega') = \frac{A(\omega,\omega')}{\sqrt{\omega\omega'}} \qquad \beta(\omega,\omega') = \frac{B(\omega,\omega')}{\sqrt{\omega\omega'}}. \qquad (71)$$

The quantity $B(\omega,\omega')$ is read off (33) (and $A(\omega,\omega')$ from the corresponding expression for $\alpha(\omega,\omega')$). From the definitions (31) and (67) of the Bogolubov coefficients, the explicit form (33) of the overlap integral and expression (22) for the field derivative one can immediately deduce that

$$B^*(\omega,\omega') = A(-\omega,\omega') \qquad A^*(\omega,\omega') = B(-\omega,\omega'). \qquad (72)$$

4. Conclusions

We considered a perfect mirror starting from rest and accelerating along the trajectory $g(t) = -\ln(\cosh t)$ until it reaches a speed arbitrarily close to c and then continuing at uniform velocity. We presented an exact evaluation of the Bogolubov amplitude $\beta(\omega,\omega')$ and showed that $|\beta(\omega,\omega')|^2$ goes as $(\omega')^{-5}$, this being a result of the velocity staying continuous throughout the trajectory and of the fact that the trajectory is asymptotically inertial. We stress that the problem examined here is not the problem studied in Fulling and Davies (1976) and Davies and Fulling (1977) where the mirror accelerates along $g(t) = -\ln(\cosh t)$ *ad infinitum*. Observe that the $\beta(\omega,\omega')$ amplitude given in Davies and Fulling (1977) cannot be obtained as the limit $r \to \ln 2$ of the amplitude evaluated in this paper. This is immediately demonstrated by the different behaviour for large ω' (in Davies and Fulling (1977) the modulus of the amplitude squared $|\beta(\omega,\omega')|^2$ goes asymptotically as $1/\omega'$). There is thus no question of the thermal spectrum arising in the case of an asymptotically inertial trajectory, regardless of how close to the speed of light the final velocity lies. The mirror trajectory $g(t) = -\ln(\cosh t)$ is of interest as a model of black hole collapse. However, a realistic mirror (or any other interface) is not expected to have its velocity forever increasing. The results of this paper can thus serve as a guide to what one should expect in the realistic case. In appendix B we construct the normal modes for a double-faced dispersive mirror accelerating to relativistic velocities. The formalism can then be applied in the calculation of the Bogolubov amplitudes in the dispersive case. This may be of relevance to the calculation of quasiparticle emission from an accelerating He3 A–B interface.

Acknowledgments

Appendix A. The trajectory $z = -\frac{1}{\kappa} \ln(\cosh \kappa t)$

A spacetime point is labelled by two coordinates (u, v) and the trajectory is specified by the function $u = f(v)$ or equally well by its inverse $v = p(u)$. From the first of (1) and the equation of the trajectory, we get after some trivial algebra

$$t = \frac{1}{2\kappa} \ln(2e^{\kappa u} - 1). \tag{73}$$

The velocity is

$$\beta(t) = \frac{\mathrm{d}z}{\mathrm{d}t} = -\tanh \kappa t$$

and

$$\sqrt{1 - \beta^2} = \frac{1}{\cosh \kappa t}.$$

The acceleration is

$$a = \frac{\mathrm{d}\beta}{\mathrm{d}t} = -\frac{\kappa}{\cosh^2 \kappa t} \tag{74}$$

and the proper acceleration is

$$\alpha = a(1 - \beta^2)^{-3/2} = -\kappa \cosh \kappa t. \tag{75}$$

Equations (6) and (73) determine $p(u)$:

$$p(u) = \frac{1}{\kappa} \ln(2e^{\kappa u} - 1) - u. \tag{76}$$

We also quote the derivative

$$p'(u) = \frac{1}{2e^{\kappa u} - 1}. \tag{77}$$

The inverse to $p(u)$ is

$$f(v) = -\frac{1}{\kappa} \ln(2 - e^{\kappa v}). \tag{78}$$

Relations (76) and (78) are valid throughout the accelerated trajectory. Obviously the trajectory asymptotically approaches the null line $l1$ (for $\kappa = 1$) (figure 3). Null lines lying above it (such as $l2$) never meet the trajectory. This is reflected in the fact that f is not real for $v \geqslant (\ln 2)/\kappa$.

To determine the velocity we use (14) to obtain

$$\beta(u) = e^{-\kappa u} - 1 \tag{79}$$

(approaching -1 as expected when $v \to \frac{\ln 2}{\kappa}$ from the left). From (15) we determine the velocity as a function of v

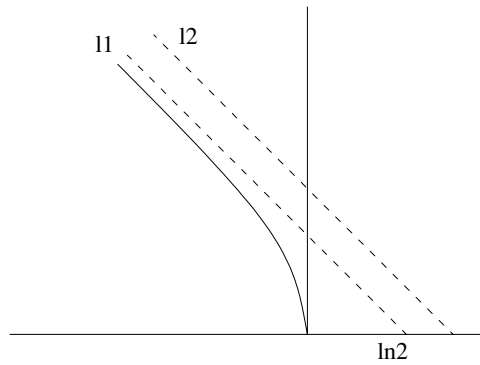$$\beta(v) = 1 - e^{\kappa v}. \tag{80}$$

Figure 3. The asymptote to the trajectory $g(t) = -\ln(\cosh t)$ and a null line lying above it.

Appendix B. Radiation from a dispersive mirror

We shall repeat the construction of section 2 in the case of a dispersive mirror. We adopt the model of Barton and Calogeracos (1995) and Calogeracos and Barton (1995). The model is described by the covariant action

$$I = \int dt \left\{ \int_{-\infty}^{\xi-} + \int_{\xi+}^{+\infty} \right\} dz \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \Phi \right\} - \gamma \int d\tau \, \Phi^2(\xi) - M \int d\tau. \tag{81}$$

The constant γ controls the dispersivity of the mirror. The mirror follows a prescribed trajectory. The statement following equation (2) remains true in the case of a dispersive mirror. Any linear combination of functions depending on u or v separately is a solution of (2). Outgoing modes will be denoted by an overbar (as in section 2) and incoming modes without an overbar. Indices $(+)$ and $(-)$ correspond to the space argument of Φ being to the right and to the left of the mirror respectively. Note that eventually the expressions for the modes will be substituted in (32). Since we evaluate the overlap integral at $t = 0$, the *in* modes to be used coincide with the usual modes for a stationary mirror as exhibited in Barton and Calogeracos (1995). The effects of the motion are manifested in the complicated appearance of the *out* modes. This is of course exactly the same state of affairs that we have already encountered in the case of a perfect mirror. The general expression for the *in* modes is given for the sake of completeness.

Incident modes (figures 4 and 5) can now come from either direction. From the right they read

$$\Phi_{(+)R\omega}(z, t) = \frac{i}{2\sqrt{\pi\omega}} \left\{ e^{-i\omega v} + R_R(\omega, p(u)) \, e^{-i\omega p(u)} \right\} \tag{82}$$

$$\Phi_{(-)R\omega}(z, t) = \frac{i}{2\sqrt{\pi\omega}} T_R(\omega, v) \, e^{-i\omega v}. \tag{83}$$

Note that each term in the above equations depends either on u or v only, hence solves (2). Similarly, a left-incident wave is written as

$$\Phi_{(+)L\omega}(z, t) = \frac{i}{2\sqrt{\pi\omega}} T_L(\omega, u) \, e^{-i\omega u} \tag{84}$$

$$\Phi_{(-)L\omega}(z, t) = \frac{i}{2\sqrt{\pi\omega}} \left\{ e^{-i\omega u} + R_L(\omega, f(v)) \, e^{-i\omega f(v)} \right\}. \tag{85}$$
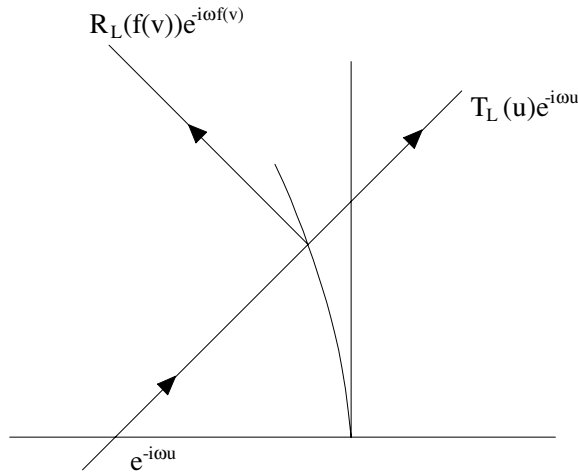
Figure 4. *In* mode with $\exp(-\mathrm{i}\omega u)$ wave incident from left, transmitted wave $T_L(u)\exp(-\mathrm{i}\omega u)$ to the right, reflected wave $R_L(f(v))\exp(-\mathrm{i}\omega f(v))$ to the left.
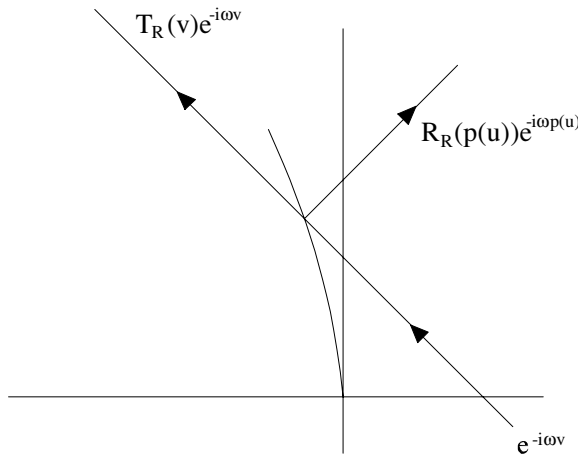

Figure 5. *In* mode with $\exp(-\mathrm{i}\omega v)$ wave incident from right, transmitted wave $T_R(v)\exp(-\mathrm{i}\omega v)$ to the left, reflected wave $R_R(p(u))\exp(-\mathrm{i}\omega p(u))$ to the right.

The above modes constitute the *in* states. The *out* modes correspond to waves transmitted to either right (figure 6) or left (figure 7). Left-transmitted modes read

$$\bar{\Phi}_{(+)R\omega}(z, t) = \frac{\mathrm{i}}{2\sqrt{\pi \omega}} \left\{ \mathrm{e}^{-\mathrm{i}\omega f(v)} + \bar{R}_R(\omega, u)\,\mathrm{e}^{-\mathrm{i}\omega u} \right\} \tag{86}$$

$$\bar{\Phi}_{(-)R\omega}(z, t) = \frac{\mathrm{i}}{2\sqrt{\pi \omega}} \bar{T}_R(\omega, f(v))\,\mathrm{e}^{-\mathrm{i}\omega v}. \tag{87}$$

The right-transmitted modes have the form

$$\bar{\Phi}_{(+)L\omega}(z, t) = \frac{\mathrm{i}}{2\sqrt{\pi \omega}} \bar{T}_L(p(u))\,\mathrm{e}^{-\mathrm{i}\omega p(u)} \tag{88}$$

$$\bar{\Phi}_{(-)L\omega}(z, t) = \frac{\mathrm{i}}{2\sqrt{\pi \omega}} \left\{ \mathrm{e}^{-\mathrm{i}\omega p(u)} + \bar{R}_L(\omega, v)\,\mathrm{e}^{-\mathrm{i}\omega v} \right\}. \tag{89}$$

$$\bar{R}_L(v)e^{-i\omega v}$$

$$\bar{T}_L(p(u))e^{i\omega p(u)}$$
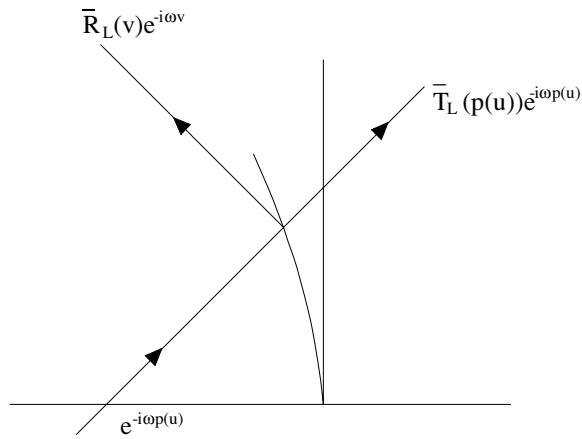
$$e^{-i\omega p(u)}$$

Figure 6. O*ut* mode with $\bar{R}_L(v)\exp(-i\omega v)$ wave to the left, $\bar{T}_L(p(u))\exp(-i\omega p(u))$ wave to the right, back-scattered wave $\exp(-i\omega p(u))$.
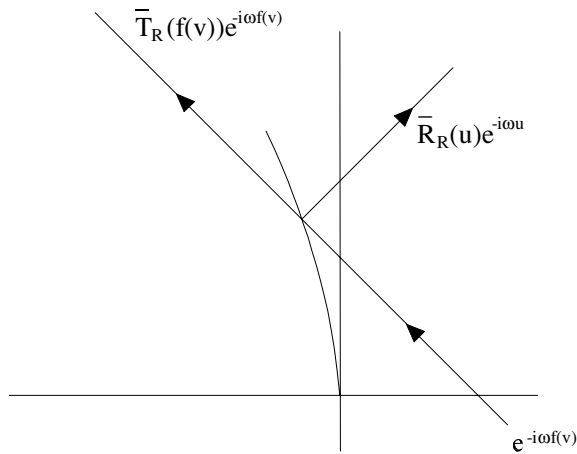
$$\bar{T}_R(f(v))e^{i\omega f(v)}$$

$$\bar{R}_R(u)e^{-i\omega u}$$

$$e^{-i\omega f(v)}$$

Figure 7. *Out* mode with $\bar{T}_R(f(v))\exp(-i\omega f(v))$ wave to the left, $\bar{R}_R(u)\exp(-i\omega u)$ wave to the right, back-scattered wave $\exp(-i\omega f(v))$.

Continuity of the modes on the trajectory is guaranteed by the trajectory equation $u = f(v)$ (9), its inverse $v = p(u)$ (8), and by

$$T = 1 + R \tag{90}$$

which is true for all the above modes due to the continuity of Φ on the trajectory (indices, overbars and arguments suppressed). The T and R for a moving mirror are nontrivial functions of spacetime and their dependence on u or v as the case may be will be determined below. We quote R and T for a mirror at rest as given in Barton and Calogeracos (1995):

$$R(\omega) = -\frac{i\gamma}{\omega + i\gamma} \qquad T(\omega) = \frac{\omega}{\omega + i\gamma}. \tag{91}$$

Concerning the mode expressions, note that the functional form of the left-incident modes (84) and (85) (figure 4) is identical to that of the right-incident modes (82) and (83) (figure 5) pertaining to a mirror moving along the trajectory $-g(t)$ (reflected with respect to the origin).

This is manifested by the fact that (84) and (85) are obtained from (83) and (82), respectively, under the correspondence $R \rightleftharpoons L$, $v \rightleftharpoons u$, $f \rightleftharpoons p$. In this respect recall the remark after (10). An analogous statement holds about the transmitted modes (86)–(89).

It should be noted that for a right-incident wave, relation (28) between the photon energy ω in the lab frame and the photon energy Ω' in the mirror's rest frame holds true. For a left-incident wave the analogue of (28) reads

$$\omega = \Omega' \frac{1-B}{\sqrt{1-B^2}}. \tag{92}$$

The discontinuity of $\partial_\mu \Phi$ across the trajectory is given in Calogeracos and Barton (1995) (cf equation (II.2) or more explicitly (III.5), after the latter has been transformed to the lab frame):

$$\mathrm{disc}\left[\beta \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} \right]_{(-)}^{(+)} = 2\gamma \Phi \sqrt{1-\beta^2} \tag{93}$$

where $\beta = \mathrm{d}g(t)/\mathrm{d}t$ is the velocity (and $z = g(t)$ the trajectory equation). The quantity Φ on the right-hand side of the above relation is evaluated on the trajectory.

We substitute the modes in the boundary condition (93), recall that the derivatives in (93) also act on R and T and use (90) to get in a straightforward manner a set of four (decoupled) differential equations, each for a set of modes.

(i) Modes transmitted to the left (equations (86) and (87)):

$$\frac{2}{\mathrm{i}\omega} \frac{\mathrm{d}\bar{R}_R(u)}{\mathrm{d}u} = -(1 + \bar{R}_R(u)) \frac{2\gamma\sqrt{1-\beta^2}}{\mathrm{i}\omega} + \bar{R}_R(u)(\beta f' - \beta + 1 + f'). \tag{94}$$

The fact that we end up with a differential equation is a direct result of the derivatives acting on R, T. To solve it we should express all quantities appearing on the right-hand side of (94) in terms of u. For a specific trajectory $g(t)$ the functions $\beta(u)$, $f'(u)$ are calculable along the lines of appendix A (at least in principle). Equation (94) is a linear first-order differential equation and a general solution may be readily written down, however, it is hardly illuminating. The study of a specific trajectory will not be pursued in the present paper.

Equation (94) for $\bar{R}_R$ may be expressed using β rather than u as the independent variable. To this end we write, using (15),

$$\beta f' - \beta + 1 + f' = 2(1 - \beta)$$

and rewrite (94) in the form

$$\frac{1}{\mathrm{i}\omega} \frac{\mathrm{d}\bar{R}_R(u)}{\mathrm{d}u} = -(1 + \bar{R}_R(u)) \frac{\gamma\sqrt{1-\beta^2}}{\mathrm{i}\omega} + \bar{R}_R(u)(1 - \beta). \tag{95}$$

We can use (19) to write the above equation in the form

$$\frac{a}{\mathrm{i}\omega(1-\beta)} \frac{\mathrm{d}\bar{R}_R(\beta)}{\mathrm{d}\beta} = -(1 + \bar{R}_R(\beta)) \frac{\gamma\sqrt{1-\beta^2}}{\mathrm{i}\omega} + \bar{R}_R(\beta)(1 - \beta). \tag{96}$$

For a specific trajectory the acceleration a may be expressed as a function of the velocity β.

As a test of the algebra and as an illustration let us consider the case of uniform motion. Then β is constant, f' is read off (24) and (94) admits the u-independent solution

$$\bar{R}_R^B(\omega) \equiv \frac{-\mathrm{i}\gamma}{\mathrm{i}\gamma + \frac{\omega(1-B)}{\gamma\sqrt{1-B^2}}}. \tag{97}$$

This is the R of (91) as expected, the quantity $\omega(1-B)/\sqrt{1-B^2}$ being the photon frequency in the frame of the mirror (cf (92)).

Consider a mirror moving at uniform velocity β_0 until it reaches the origin O, accelerates from O to P and then continues at uniform velocity β_P. The differential equation must then be solved from $\beta = \beta_0$ to $\beta = \beta_P$ with initial condition

$$\bar{R}_R(\omega, \beta) = \bar{R}_R^{\beta_0}(\omega) \tag{98}$$

where the right-hand side of (98) is given by (97). At point P the solution of the differential equation joins smoothly with $\bar{R}_R^{\beta_P}(\omega)$.

(ii) Modes transmitted to the right (equations (88) and (89)):

$$\frac{2}{i\omega} \frac{d\bar{R}_L(v)}{dv} = (1 + \bar{R}_L(v)) \frac{2\gamma\sqrt{1-\beta^2}}{i\omega} + \bar{R}_L(v)(\beta p' - p' - \beta - 1). \tag{99}$$

Now everything on the right-hand side should be expressed in terms of v. This again is in principle straightforward once the equation of the trajectory is given.

We can repeat the analysis of (i) above. We use (14) to write

$$\beta p' - p' - \beta - 1 = -2(1 + \beta)$$

and rewrite (99) in the form

$$\frac{1}{i\omega} \frac{d\bar{R}_L(v)}{dv} = (1 + \bar{R}_L(v)) \frac{\gamma\sqrt{1-\beta^2}}{i\omega} - \bar{R}_L(v)(1 + \beta). \tag{100}$$

We use (20) to rewrite the above in the form

$$\frac{a}{i\omega(1+\beta)} \frac{d\bar{R}_L(\beta)}{d\beta} = (1 + \bar{R}_L(\beta)) \frac{\gamma\sqrt{1-\beta^2}}{i\omega} - \bar{R}_L(\beta)(1 + \beta). \tag{101}$$

Concerning the solution of (101), the reader is referred to the remarks made following (96). In the case of uniform motion we obtain the solution in a similar way as before

$$\bar{R}_L^B(\omega) \equiv \frac{-i\gamma}{i\gamma + \frac{\omega(1+B)}{\gamma\sqrt{1-B^2}}}. \tag{102}$$

Note that in the denominator we again have the frequency in the mirror's rest frame. Note also the expected difference in the sign of β between equations (97) and (102). This is due to the fact that the transmitted wave is now in the opposite direction than before. Similarly, we obtain the differential equations for the last two classes of modes.

(iii) Modes incident from the left (equations (84) and (85)):

$$\frac{2}{i\omega} \frac{dT_L(u)}{du} = -T_L(u) \frac{2\gamma\sqrt{1-\beta^2}}{i\omega} + T_L(u)(\beta f' - \beta + 1 + f') - (\beta f' - \beta + 1 + f'). \tag{103}$$

(iv) Modes incident from the right (equations (82) and (83)):

$$\frac{2}{i\omega} \frac{dT_R(v)}{dv} = -T_R(v) \frac{2\gamma\sqrt{1-\beta^2}}{i\omega} + T_R(v)(\beta p' - p' - \beta - 1) - (\beta p' - p' - \beta - 1). \tag{104}$$

Equations (103) and (104) can be cast in terms of a and β via the same manipulations used in obtaining (96) and (101). The four uncoupled linear first-order differential equations (94), (99), (103) and (104) may be solved (at least to some approximation) once the form of the trajectory is specified.

After we obtain the solutions, or an approximation to them, we can use the formalism of the preceding sections to calculate the Bogolubov amplitudes. There are some obvious changes. The indices i, j appearing in (29) and (30) now stand for both the frequency ω and the direction of incidence (R or L). The emission amplitude $\beta(\omega_{L,R}, \omega'_{L,R})$ is given by the standard expression (32) where now the integration ranges from $z = -\infty$ to ∞.

The covariant model of a dispersive mirror allows estimates about the spectrum of the emitted radiation without solving the above differential equations. The amplitude $\beta(\omega, \omega')$ calculated in the previous section in the context of perfect mirrors corresponds to $\beta(\omega_R, \omega'_R)$ in the notation of this section. If we describe the photons in the rest frame of the mirror then, roughly speaking, frequencies less than γ are reflected whereas those greater than γ are transmitted (this is no more than a very crude qualitative estimate resulting from (91)). According to the treatment in the preceding sections ω' labels the *in* states where the mirror is at rest. Thus the *in* frequencies that roughly matter are in the range $\omega' < \gamma$. The frequency ω labels an *out* state and is connected to the frequency Ω' in the mirror's rest frame via (28) or (92) depending on the direction of incidence, both essentially amounting to a Lorentz transformation from one frame to another. For a receding mirror and right-incident photons the two frequencies are connected in terms of β_P (the velocity at *P)*

$$\Omega' = \omega \sqrt{\frac{1 + |\beta_P|}{1 - |\beta_P|}}. \tag{105}$$

Hence the *out* frequencies that matter for $\beta(\omega_R, \omega'_R)$ are in the range

$$\omega < \gamma \sqrt{\frac{1 - |\beta_P|}{1 + |\beta_P|}}. \tag{106}$$

This limit is easily saturated in the case of high terminal velocity β_P. This of course does not apply to the amplitude $\beta(\omega'_L, \omega_L)$ for a mirror executing the same motion where the signs in (105) and (106) are interchanged between numerator and denominator. Having made these qualitative remarks, one must be warned that the role of high frequencies (compared to γ) in the context of dispersive mirrors should not be underestimated; experience has shown that a finite γ does not prevent integrations from becoming ultraviolet divergent.

Note added. After the completion of this work the paper by Nicolaevici (2001) appeared where the model of Barton and Calogeracos (1995) is used in a similar way to obtain the differential equations for the *T* and *R* amplitudes. The author focuses on the calculation of local field quantities rather than on the calculation of Bogolubov amplitudes.

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